1 (a) For any $k \geq 1$, since $A_k \in \mathcal{M}$, χ_A is measurable.

Therefore, $f := \sum_{k=1}^{\infty} \chi_{A_k}$ is measurable.

Hence, $E = f^{-1}(\{2024\})$ is measurable.

(b) Since $\sum_{k=1}^{\infty} \lambda(A_k) < \infty$, $\sum_{k=n}^{\infty} \lambda(A_k) \to 0$ as $n \to \infty$.

For any $n \in \mathbb{N}$, $F \subset \bigcup_{k \geq n} A_k$.

Thus $\lambda(F) \leq \lambda(\bigcup_{k \geq n} A_k) \leq \sum_{k=n}^{\infty} \lambda(A_k) \to 0$ as $n \to \infty$.

$\square$

2 (a) It suffices to show

$$(f+g)^{-1}(a,\infty) = \bigcup_{\substack{t+s>a \\ t,s\in\mathbb{Q}}} f^{-1}(t,\infty) \cap g^{-1}(s,\infty).$$

Consider a point x satisfying $(f+g)(x) > a$.

We can always choose $t, s \in \mathbb{Q}$ such that $f(x) > t$, $g(x) > s$. It follows that $x \in f^{-1}(t,\infty) \cap g^{-1}(s,\infty)$, and we have one side inclusion. The other side is immediately.

Since $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$, it suffices to show f^2 is measurable if f is measurable. It follows the fact $h(x) = x^2$ is a continuous function.

(b) Let $\tilde{\mathcal{F}} := \{E \in P_{\mathbb{R}} : f(E) \in \mathcal{B}\}$.

We first show $\tilde{\mathcal{F}}$ is a σ-algebra.

Since f is continuous and injective, f is monotone.

- $\mathbb{R} \in \tilde{\mathcal{F}}$ since $f(\mathbb{R})$ is an interval
- If $E \in \tilde{\mathcal{F}}$, then $f(E) \in \mathcal{B}$.
  $f(\mathbb{R} \setminus E) = f(\mathbb{R}) \setminus f(E) \in \mathcal{B}$.
  Thus $\mathbb{R} \setminus E \in \tilde{\mathcal{F}}$.
- If $E_k \in \tilde{\mathcal{F}}$, then $f(E_k) \in \mathcal{B}$.
  $f(\bigcup_{k=1}^{\infty} E_k) = \bigcup_{k=1}^{\infty} f(E_k) \in \mathcal{B}$.

Since f maps compact sets to compact sets, $\tilde{\mathcal{F}}$ contains all compact sets.

Hence, $\mathcal{B} \subset \tilde{\mathcal{F}}$, i.e., $f(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$.

$\square$

3. Assume on the contrary there is $\varepsilon_0 > 0$ and E_n with $\mu(E_n) \leq 2^{-n}$ such that $\int_{E_n} |f| \, d\mu \geq \varepsilon_0$.

Let $A_n = \bigcup_{j \geq n} E_j$ and $A = \bigcap_{n=1}^{\infty} A_n$.

By Borel-Cantelli Lemma, since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, $\mu(A) = 0$.

On the other hand, since $|f| \chi_{A_n} \leq |f|$, by Dominated Convergence Theorem,

$$\int_A |f| \, d\mu = \lim_{n \to \infty} \int_{A_n} |f| \, d\mu \geq \varepsilon_0 .$$

Contradiction!

4(a) Suppose not.

Then for any $n \in \mathbb{N}$, $\exists$ measurable $A_n \subset B$ such that $\mathcal{L}(B \setminus A_n) \leq \frac{1}{n}$.

Let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is measurable.

And $\mathcal{L}(B \setminus A) \leq \mathcal{L}(B \setminus A_n) \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$.

Thus $\mathcal{L}(B \setminus A) = 0$. Therefore, $B \setminus A$ is measurable.

Hence, $B = (B \setminus A) \cup A$ is measurable.

Contradiction!

(b) Notice that any Lebesgue measurable set

$$E = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup N \text{ with } K_n \text{ compact and } \mathcal{L}(N)=0$$

and continuous function maps compact sets to compact sets.

It suffices to show f preserves null set.

Let $N_n = N \cap [-n, n]$.

Then $N = \bigcup_{n=1}^{\infty} N_n$.

Since f is continuously differentiable, f is Lipschitz on $[-n,n]$, i.e., $\exists$ $M_n > 0$ such that

$|f(x) - f(y)| < M_n |x-y|$ for all $x, y \in [-n, n]$.

Thus $\mathcal{L}(f(N_n)) = 0$.

Hence, $\mathcal{L}(f(N)) = \mathcal{L}\left(\bigcup_{n=1}^{\infty} f(N_n)\right) \leq \sum_{n=1}^{\infty} \mathcal{L}(f(N_n)) = 0$.

$\square$

5. The statement is wrong.

Counter-example:

$f_1 = \chi_{[0, 1/2]}$

$f_2 = \chi_{[1/2, 1]}$

$f_3 = \chi_{[0, 1/2^2]}$

$f_4 = \chi_{[1/2^2, 1/2]}$

$f_5 = \chi_{[1/2, 3/2^2]}$

$f_6 = \chi_{[3/2^2, 1]}$

$\vdots$